HEURISTIC DERIVATION OF BLACKBODY RADIATION LAWS USING PRINCIPLES OF DIMENSIONAL ANALYSIS

Gerhard Kramm

Geophysical Institute, University of Alaska Fairbanks
903 Koyukuk Drive Fairbanks, AK 99775-7320, USA

Fritz Herbert

J.W. Goethe-University, Theoretical Meteorology
Robert-Mayer-Strasse 1, D-60325 Frankfurt am Main, Germany

Abstract

A generalized form of Wien's displacement law and the blackbody radiation laws of (a) Rayleigh and Jeans, (b) Rayleigh, (c) Wien and Paschen, (d) Thiesen and (e) Planck are derived using principles of dimensional analysis. This kind of scaling is expressed in a strictly mathematical manner employing dimensional π-invariants analysis sometimes called Buckingham's π-theorem. It is shown that in the case of the classical radiation law of Rayleigh and Jeans only one π number occurs that has to be considered as a non-dimensional universal constant. This π number may be determined theoretically or/and empirically. It is also shown that dimensional π-invariants analysis yields a generalized form of Wien's displacement law. In this instance two π numbers generally occur. Consequently, a universal function is established that is indispensable to avoid the so-called *Rayleigh-Jeans catastrophe in the ultraviolet*. Unfortunately, such a universal function cannot be inferred from dimensional arguments. It has to be derived theoretically or/and empirically, too. It is shown that such a similarity function can be deduced on the basis of heuristic principles, when criteria like the maximum condition regarding the generalized form of Wien's displacement law, the requirement of the power law of Stefan and Boltzmann, and Ehrenfest's arguments regarding the *red* and the *violet requirements* are adopted.

1   Introduction

Our contribution is focused on the heuristic derivation of blackbody radiation laws, where principles of dimensional analysis are considered. The idea on which dimensional analysis is based is very simple. It is inferred from the fact that physical laws do not depend on arbitrarily chosen basic units of measurements. In recognizing this simple idea, one may conclude that the functions that express physical laws must possess a certain fundamental property, which, from a mathematical point of view, is called the generalized homogeneity or symmetry [3]. This property allows the number of arguments in these functions to be reduced, thereby making it simpler to obtain them. As Barenblatt [3] pointed out, this is the entire content of dimensional analysis - there is nothing more to it.

Often, solutions for physical problems, especially in mechanics and fluid mechanics (e.g., [1, 2, 3, 4, 5, 18, 23, 26, 49]) and cloud microphysics [9], can be found on the basis of similarity hypotheses that comprise all problem-relevant dimensional quantities and serve to possess the physical mechanisms of these problems. Such similarity hypotheses implicitly describe the functional dependence between these dimensional quantities in a mathematical form. This does not mean that this functional dependence can explicitly substantiated by formulating a similarity hypothesis only. A similarity hypothesis will become successful if a generalize

homogeneity or a symmetry exist.

If similarity is hypothesized, its mathematical treatment can further be performed by the procedure of dimensional π-invariants analysis (sometimes also called the Buckingham's π-theorem, for details see [1, 2, 3, 4, 10, 26]). During this mathematical treatment the explicit dependence between the problem-relevant dimensional quantities is expressed by the non-dimensional π-invariants.

The first attempt to derive a blackbody radiation law on the basis of dimensional consideration was performed by Jeans [19, 21] using the wavelength λ, the absolute temperature T, the velocity of light in vacuum c, the charge e and the mass m of an electron, the universal gas constant R, and the dielectric constant of the ether K. His attempt to derive Wien's [47] displacement law, however, was strongly criticized by Ehrenfest [12, 13]. Ehrenfest showed that this kind of dimensional analysis which leads to a similarity function containing two π numbers (see Appendix A) was rather arbitrary. In his reply Jeans [21] rejected Ehrenfest's criticisms. Some months later, Ehrenfest [13] also plucked Jeans' additional arguments to pieces. Since their debate was mainly focus on dimensional analysis for which a closed theory was not available during that time (a first step on this road was made by Buckingham [10]), there was, if at all, only a minor interest on this debate [33].

In the following we will derive a generalized form of Wien's [47] displacement law and various blackbody radiation laws from the perspective of dimensional scaling. In doing so, we partly follow the ideas outlined by Glaser (as cited by [43]) and Sommerfeld [43], i.e., we only postulate that similarity exists. In the chapter 2, the method used in our dimensional scaling is presented in a strict mathematical manner. The application of our method in deriving the blackbody radiation laws is described in chapter 3. Here, it is shown that in the case of the classical radiation law of Rayleigh [29, 30] and Jeans [20] only one π number occurs that has to be considered as a non-dimensional universal constant. This π number may be determined theoretically or/and empirically. In the instance of the generalized form of Wien's [47] displacement law that, in principle, contains the radiation laws of Wien [48] and Paschen [36], Rayleigh [29], Thiesen [46] and Planck [37, 38, 39] as special cases, two π numbers generally occur. Consequently, a universal function (also called the similarity function) is established that is indispensable to avoid the so-called *Rayleigh-Jeans catastrophe in the ultraviolet* [14]. Unfortunately, such a universal function cannot explicitly be determined by dimensional π-invariants analysis. It has to derive on the basis of theoretical or/and empirical findings, too. It is shown, however, that such a similarity function can be deduced on the basis of heuristic principles, when criteria like the maximum condition regarding the generalized form of Wien's [47] displacement law, the requirement of the power law of Stefan [45] and Boltzmann [7], and Ehrenfest's [14] *red* and the *violet requirements* are adopted. Since even Planck's radiation law can be derived in such a manner, some historical notes regarding the foundation of the quantum theory are briefly gathered in chapter 4.

2 Dimensional π-invariants analysis

The theoretical foundation of the procedure is linked to various sources, for instance, Kitaigorodskij [23], Barenblatt [1, 2, 3, 4], Herbert [17], Pal Arya [35], Brown [9], Sorbjan [44], and Kramm and Herbert [26] which are devoted to characteristic scaling problems in fluid dynamics and turbulence, boundary layer meteorology and other physical disciplines. The description mainly follows the guideline of Kramm and Herbert [26].

Let adopt that, associated with a certain physical problem, we can select a set of characteristic dimensionality quantities, for instance, κ variables, parameters or/and constants, $Q_1, Q_2, \ldots, Q_\kappa$ that unambiguously and evidently represent the arguments of a mathematical relationship. First this "law" is unspecified; therefore it is formally employed as a general postulate, commonly referred to as the similarity hypothesis of the problem, which may read

$$F(Q_1, Q_2, \ldots, Q_\kappa) = 0 \quad . \tag{2.1}$$

In its implicit representation Eq. (2.1) declares $\kappa - 1$ free or independent arguments as well as a transformation of the full series of Q_j for $j = 1, \ldots, \kappa$ to a series of p non-dimensional invariants π_i for $i = 1, \ldots, p$

in terms of a factorization by powers. Correspondingly, in that mind each π_i-expression is defined by

$$\pi_i = Q_1^{x_{1,i}} Q_2^{x_{2,i}} \dots Q_\kappa^{x_{\kappa,i}} = \prod_{j=1}^{\kappa} Q_j^{x_{j,i}} \quad \text{for } i = 1, \dots, p \quad , \tag{2.2}$$

and it is necessarily linked with the condition of non-dimensionality

$$\dim \pi_i = 1 \quad \text{for } i = 1, \dots, p \quad , \tag{2.3}$$

where $p < \kappa$ is customarily valid.

Next, we will suppose that the π-invariants can have interdependencies of arbitrary forms, and it may exist a corresponding relation

$$\phi(\pi_1, \pi_2, \dots, \pi_p) = 0 \tag{2.4}$$

which is to be understood as an alternative similarity hypothesis to Eq. (2.1). In this function the powers $x_{j,i}$ are basically unknown numbers, and their determination is the proper problem of the so-called Buckingham π-theorem. If there are more than one π-invariant, i.e., $p > 1$, then we have with Eq. (2.4) the explicit representation

$$\pi_i = \varphi(\pi_1, \pi_2, \dots, \pi_p) \tag{2.5}$$

in which φ may be interpreted as a universal function within the framework of the similarity hypothesis, where, according to the implicit formulation (2.4), π_i (for any arbitrary $i \in \{1, \dots, p\}$) is not an argument of that universal function. Note that in the special case of $p = 1$, we will merely obtain one π-invariant, that is a non-dimensional universal constant. This special case is expressed by Eq. (2.5) in the singular form

$$\pi = \text{const.} \tag{2.6}$$

(or $\varphi = \text{const.}$). In view to the determination of the powers $x_{j,i}$, we will extend our treatment to the concise set of *fundamental* dimensions, D_n for $n = 1, \dots, r$, such as length L, time T, mass M, temperature Θ, considering that any quantity's dimension can be analysed in terms of the independent D_n by homogeneous power factorization. Let that be expressed as

$$\dim Q_j = D_1^{g_{1,j}} D_2^{g_{2,j}} \dots D_r^{g_{r,j}} = \prod_{n=1}^{r} D_n^{g_{n,j}} \quad \text{for } j = 1, \dots, \kappa \quad , \tag{2.7}$$

in which the powers $g_{n,j}$ for $n = 1, \dots, r$ and $j = 1, \dots, \kappa$ are known from the relevant quantities Q_j according to the hypothesized similarity condition. Note that $r \le \kappa$ is valid, where r is the highest number of fundamental dimensions that may occur. In other words: for κ quantities Q_j including r fundamental dimensions D_n we obtain $p = \kappa - r$ independent non-dimensional invariants, so-called π numbers.

Now a straight-forward development of the analytical framework is attained by introducing Eq. (2.7) together with the factorization by powers from Eq. (2.2) into the condition of non-dimensionality (2.3). In doing so, we obtain this basic law as described in the following detailed representation

$$\dim \pi_i = \prod_{j=1}^{\kappa} \Big(\prod_{n=1}^{r} D_n^{g_{n,j}} \Big)^{x_{j,i}} = 1 \quad \text{for } i = 1, \dots, p \quad . \tag{2.8}$$

Combining the two factorizations $\prod_j$ and $\prod_n$ in this equation enables to rewrite this set of conditions in the fully equivalent form

$$\dim \pi_i = \prod_{n=1}^{r} D_n^{\sum_{j=1}^{\kappa} g_{n,j} \, x_{j,i}} = 1 \quad \text{for } i = 1, \dots, p \quad . \tag{2.9}$$

For the following conclusion, the latter is more suitable than the former. Indeed, we may immediately infer from the factorizing analysis in dependence on the bases D_n for $n = 1, \ldots, r$, that the set of condition

$$\sum_{j=1}^{\kappa} g_{n,j} \, x_{j,i} = 0 \quad \text{for } n = 1, \ldots, r \text{ and } i = 1, \ldots, p \quad , \tag{2.10}$$

has to hold since each D_n-exponential factor must satisfy, owing to its mathematical independence, the condition of non-dimensionality (see Eqs. (2.3) and (2.9)), i.e., to be equal to unity. In matrix notation, Eq. (2.10) may be expressed by

$$\underbrace{\left\{ \begin{array}{cccc} g_{1,1} & g_{1,2} & \cdots & g_{1,\kappa} \\ g_{2,1} & g_{2,2} & \cdots & g_{2,\kappa} \\ \cdots & \cdots & \cdots & \cdots \\ g_{r,1} & g_{r,2} & \cdots & g_{r,\kappa} \end{array} \right\}}_{\substack{\text{dimensional matrix} \\ \mathbf{G} = \{g_{r,\kappa}\}}} \underbrace{\left\{ \begin{array}{cccc} x_{1,1} & x_{1,2} & \cdots & x_{1,p} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,p} \\ \cdots & \cdots & \cdots & \cdots \\ x_{\kappa,1} & x_{\kappa,2} & \cdots & x_{\kappa,p} \end{array} \right\}}_{\substack{\text{matrix of powers} \\ \mathbf{A} = \{x_{\kappa,p}\}}} = \{0\} \quad , \tag{2.11}$$

where the notation $\{0\}$ is an $r \times p$ matrix, and each column of the matrix of powers, $\mathbf{A}$, is forming so-called solution vectors $\mathbf{x}_i$ for the invariants π_i for $i = 1, \ldots, p$. The set of equations (2.11) serves to determine the powers $x_{j,i}$ for $j = 1, \ldots, \kappa$, and $i = 1. \ldots, p$. So the homogeneous system of linear equations has, in accord with Eq. (2.11), for each of these π-invariants the alternative notation

$$\mathbf{G} \cdot \mathbf{x}_i = \mathbf{0} \quad \text{or} \quad \left\{ \begin{array}{cccc} g_{1,1} & g_{1,2} & \cdots & g_{1,\kappa} \\ g_{2,1} & g_{2,2} & \cdots & g_{2,\kappa} \\ \cdots & \cdots & \cdots & \cdots \\ g_{r,1} & g_{r,2} & \cdots & g_{r,\kappa} \end{array} \right\} \left\{ \begin{array}{c} x_{1,i} \\ x_{2,i} \\ \cdots \\ x_{\kappa,i} \end{array} \right\} = \{0\} \quad \text{for } i = 1, \ldots, p \quad . \tag{2.12}$$

The rank of the dimensional matrix is equal to the number of fundamental dimensions, r. If the number of dimensional quantities, κ, is equal to r, we will obtain: $p = 0$. In this case there is only a trivial solution. In the case of $p > 0$, the homogeneous system of linear equation (2.12) is indeterminate, i.e., more unknowns than equations, a fact that is true in all instances presented here. Hence, for each of the p non-dimensional π numbers, it is necessary to make a reasonable choice for p of these unknowns, $x_{\kappa,i}$, to put this set of equations into a solvable state. After that we obtain for each π number an inhomogeneous linear equation system that serves to determine the remaining $r = \kappa - p$ unknowns. Thus, the remaining $r \times r$ dimensional matrix $\mathbf{G}_0 = \{g_{r,r}\}$ has the rank r, too. It is the largest square sub-matrix for which the determinant is unequal to zero ($|g_{r,r}| \neq 0$). Thus, we have

$$\mathbf{G}_0 \cdot \mathbf{x}_i = \mathbf{B}_i \quad \text{or} \quad \left\{ \begin{array}{cccc} g_{1,1} & g_{1,2} & \cdots & g_{1,r} \\ g_{2,1} & g_{2,2} & \cdots & g_{2,r} \\ \cdots & \cdots & \cdots & \cdots \\ g_{r,1} & g_{r,2} & \cdots & g_{r,r} \end{array} \right\} \left\{ \begin{array}{c} x_{1,i} \\ x_{2,i} \\ \cdots \\ x_{r,i} \end{array} \right\} = \left\{ \begin{array}{c} B_{1,i} \\ B_{2,i} \\ \cdots \\ B_{r,i} \end{array} \right\} \quad \text{for } i = 1, \ldots, p \quad . \tag{2.13}$$

This inhomogeneous system of linear equations can be solved for $x_{m,i}$ for $m = 1, \ldots, r$ by employing Cramer's rule.

3   Similarity hypotheses for blackbody radiation laws

In 1860 Kirchhoff [22] proposed his famous theorem that for any body at a given temperature the ratio of emissivity, e_ν, i.e., the intensity of the emitted radiation at a given frequency ν, and absorptivity, a_ν, of the radiation of the same frequency is the same generally expressed by

$$\frac{e_\nu}{a_\nu} = J(\nu, T) \quad , \tag{3.1}$$

where he called a body perfectly black when $a_\nu = 1$ so that $J(\nu, T)$ is the emissive power of such a black body [22, 34]. An example of a perfectly black body is the 'Hohlraumstrahlung' that describes the radiation in a cavity bounded by any emitting and absorbing substances of uniform temperature which are opaque. The state of the thermal radiation which takes place in this cavity is entirely independent of the nature and properties of these substances and only depends on the absolute temperature, T, and the frequency (or the wavelength λ or the angular frequency $\omega = 2\pi\nu$). For this special case the radiation is homogeneous, isotropic and unpolarized so that we have [34]

$$J(\nu, T) = \frac{c}{8\pi} U(\nu, T) \quad .$$
(3.2)

Here, $U(\nu, T)$ is called the monochromatic (or spectral) energy density of the radiation in the cavity. Kirchhoff's theorem has become one of the most general of radiation theory and expresses the existence of temperature equilibrium for radiation, as already pointed out by Wilhelm Wien during his Nobel Lecture given in 1911. Expressions for the monochromatic energy density called the blackbody radiation laws were derived by (a) Rayleigh [29, 30] and Jeans [20], (b) Wien [48] and Paschen [36], (c) Thiesen [46], (d) Rayleigh [29], and (e) Planck [37, 38, 39].

3.1 The Rayleigh-Jeans law

The dimensional π-invariants analysis described in chapter 2 can be employed to derive all these blackbody radiation laws. First, we consider the radiation law of Rayleigh [29, 30] and Jeans [20] for which the similarity hypothesis $F(Q_1, Q_2, Q_3, Q_4, Q_5) = F(U, \nu, T, c, k)$ is postulated. Here, $Q_1 = U$, $Q_2 = \nu$, $Q_3 = T$, $Q_4 = c$, the velocity of light in vacuum, and $Q_5 = k$, the Boltzmann constant. Obviously, the number of dimensional quantities is $\kappa = 5$. The dimensional matrix is given by

$$\mathbf{G} = \left\{ \begin{array}{ccccc} -1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ -1 & -1 & 0 & -1 & -2 \\ 1 & 0 & 0 & 0 & 1 \end{array} \right\}$$
(3.3)

that can be derived from the table of fundamental dimensions,

	U	ν	T	c	k
Length	- 1	0	0	1	2
Temperature	0	0	1	0	- 1
Time	- 1	- 1	0	- 1	- 2
Mass	1	0	0	0	1

In accord with Eq. (2.12), the homogeneous system of linear equations is given by

$$\left\{ \begin{array}{ccccc} -1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ -1 & -1 & 0 & -1 & -2 \\ 1 & 0 & 0 & 0 & 1 \end{array} \right\} \left\{ \begin{array}{c} x_{1,1} \\ x_{2,1} \\ x_{3,1} \\ x_{4,1} \\ x_{5,1} \end{array} \right\} = \{0\}$$
(3.4)

Obviously, the rank of the dimensional matrix is $r = 4$, and we have $p = \kappa - r = 1$ non-dimensional π number that can be deduced from

$$\left. \begin{array}{rcl} -x_{1,1} \qquad\qquad\qquad +x_{4,1} + 2\,x_{5,1} &=& 0 \\ x_{3,1} \qquad\qquad - x_{5,1} &=& 0 \\ -x_{1,1} - x_{2,1} \qquad - x_{4,1} - 2\,x_{5,1} &=& 0 \\ x_{1,1} \qquad\qquad\qquad + x_{5,1} &=& 0 \end{array} \right\} \quad .$$
(3.5)

This equation set results in $x_{2,1} = -2\,x_{1,1}$, $x_{3,1} = -x_{1,1}$, $x_{4,1} = 3\,x_{1,1}$, and $x_{5,1} = -x_{1,1}$. Choosing $x_{1,1} = 1$[1] yields (see Eq. (3.2))

$$\pi_1 = \prod_{j=1}^{5} Q_j^{x_{j,1}} = U^1\,\nu^{-2}\,T^{-1}\,c^3\,k^{-1} \quad . \tag{3.6}$$

Rearranging provides finally

$$U(\nu, T) = \pi_1 \frac{\nu^2}{c^3}\,k\,T \quad , \tag{3.7}$$

where the π-invariant can be identified as $\pi_1 = 8\pi$. Thus, we have

$$U(\nu, T) = \frac{8\,\pi\,\nu^2}{c^3}\,k\,T \quad . \tag{3.8}$$

This radiation law was first derived by Rayleigh [29, 30] using principles of classical statistics, with a correction by Jeans [20]. Lorentz [31] derived it in a somewhat different way. Obviously, it fulfills both Kirchhoff's [22] findings and the requirements of Wien's [47] conventional displacement law, $U(\nu, T) \propto \nu^3 f(\nu, T)$. Equation (3.8) is called the classical blackbody radiation law because, as already pointed out by Sommerfeld [43], it is restricted to the two constants c and k which are well known in classical physics. In the case when the frequency is small and the temperature relatively high, this formula works well (indeed it can be considered as an asymptotic solution for $\nu \to 0$), as experimentally proofed by Lummer and Pringsheim [32] and especially by Rubens and Kurlbaum [41, 42]. But it is obvious that the law of Rayleigh [29, 30] and Jeans [20] cannot be correct because for $\nu \to \infty$ the monochromatic energy density, $U(\nu, T)$, would tend to infinity. Ehrenfest [14] coined it the *Rayleigh-Jeans catastrophe in the ultraviolet*. Consequently, the integral over $U(\nu, T)$ for all frequencies yielding the energy density,

$$E(T) = \int_0^\infty U(\nu, T)\,d\nu = \frac{8\,\pi}{c^3} k\,T \int_0^\infty \nu^2\,d\nu = \infty \quad , \tag{3.9}$$

would become divergent [15], in complete contrast to Boltzmann's [7] thermodynamic derivation of the T^4 law,

$$E(T) = a\,T^4 \tag{3.10}$$

and Stefan's [45] empirical finding

$$R(T) = \frac{c}{4} E(T) = \frac{c\,a}{4}\,T^4 = \sigma\,T^4 \tag{3.11}$$

where $\sigma = c\,a/4$ is customarily called the Stefan constant.

3.2   A generalized form of Wien's displacement law

Since this classical blackbody radiation law completely fails in the case of high frequencies, it is necessary to look for an improved similarity hypothesis that must lead to more than one π-invariant so that a universal function (or similarity function) is established. Such a universal function is indispensable to ensure that the integral in Eq. (3.9) stays finite. However, we cannot introduce another independent variable to get more than one π-invariant because it would be in disagreement with Kirchhoff's [22] findings. Since, on the other hand, the Rayleigh-Jeans law seems to be reasonable for small frequencies, one may state that the two dimensional

---

[1]) If we choose $x_{1,1} = \alpha$, where $\alpha \neq 0$ is an arbitrary real number, we will lead to another invariant π_1^*. The relationship between this π-invariant and that, occurring in Eq. (3.6), is given by $\pi_1 = \sqrt[\alpha]{\pi_1^*}$. Therefore, for convenience, we may simply choose: $x_{1,1} = 1$.

constants of the classical physics, c and k, should also occur in an improved similarity hypothesis. Consequently, we must look for another dimensional constant [43] and have to make a choice for its fundamental dimensions. Thus, an improved similarity hypothesis reads $F(Q_1, Q_2, Q_3, Q_4, Q_5, Q_6) = F(U, \nu, T, c, k, \eta) = 0$, i.e., instead of the restriction to the two classical constants c and k, we now have three dimensional constants in our similarity hypothesis, where the dimensions of the third constant, $Q_6 = \eta$, is to be chosen in such a sense that $\eta \, \nu/(k \, T)$ becomes non-dimensional, in hoping to establish another π-invariant, and, hence, a universal function $\varphi_R(\eta \, \nu/(k \, T))$, according to the explicit representation $\pi_1 = \varphi_R(\pi_2)$ expressed by Eq. (2.5). This means that not only $k \, T$, but also $\eta \, \nu$ have the dimensions of energy. Obviously, the number of dimensional quantities is $\kappa = 6$, and the table of fundamental dimensions reads

	U	ν	T	c	k	η
Length	- 1	0	0	1	2	2
Temperature	0	0	1	0	- 1	0
Time	- 1	- 1	0	- 1	- 2	- 1
Mass	1	0	0	0	1	1

leading to the dimensional matrix:

$$\mathbf{G} = \begin{Bmatrix} -1 & 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ -1 & -1 & 0 & -1 & -2 & -1 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{Bmatrix} \quad . \tag{3.12}$$

The homogeneous system of linear equations can then be expressed by (see Eq. (2.12))

$$\begin{Bmatrix} -1 & 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ -1 & -1 & 0 & -1 & -2 & -1 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{Bmatrix} \begin{Bmatrix} x_{1,i} \\ x_{2,i} \\ x_{3,i} \\ x_{4,i} \\ x_{5,i} \\ x_{6,i} \end{Bmatrix} = \{0\} \quad \text{for } i = 1, 2 \tag{3.13}$$

where the rank of the dimensional matrix is $r = 4$, i.e., we have $p = \kappa - r = 2$ non-dimensional π-invariants so that a universal function, as urgently required, is established. These two π numbers can be derived from

$$\left. \begin{aligned} - x_{1,i} && + x_{4,i} & + 2 \, x_{5,i} & + 2 \, x_{6,i} & = 0 \\ & x_{3,i} && - x_{5,i} && = 0 \\ - x_{1,i} & - x_{2,i} & - x_{4,i} & - 2 \, x_{5,i} & - x_{6,i} & = 0 \\ x_{1,i} &&& + x_{5,i} & + x_{6,i} & = 0 \end{aligned} \right\} \quad \text{for } i = 1, 2 \quad . \tag{3.14}$$

Choosing $x_{1,1} = 1$, $x_{6,1} = N$, $x_{1,2} = 0$, and $x_{6,2} = 1$ yields $x_{2,1} = -2 + N$, $x_{3,1} = -1 - N$, $x_{4,1} = 3$, $x_{5,1} = -1 - N$, $x_{2,2} = 1$, $x_{3,2} = -1$, $x_{4,2} = 0$, and $x_{5,2} = -1$. Note that the choice $x_{1,1} = 1$ and $x_{1,2} = 0$ is indispensable to ensure that $U(\nu, T)$ only occurs explicitly. Furthermore, $N < 3$ is a real number, and $x_{6,2} = 1$ is chosen in such a sense that $\eta \, \nu/(k \, T)$ is established as a π-invariant. According to Eq. (2.2), the π-invariants are then given by

$$\pi_1 = \prod_{j=1}^{6} Q_j^{x_{j,1}} = U^1 \, \nu^{-2+N} \, T^{-1-N} \, c^3 \, k^{-1-N} \, \eta^N = \frac{U \, c^3}{\nu^2 \, k \, T} \left(\frac{\eta \, \nu}{k \, T} \right)^N \tag{3.15}$$

and

$$\pi_2 = \prod_{j=1}^{6} Q_j^{x_{j,2}} = U^0 \, \nu^1 \, T^{-1} \, c^0 \, k^{-1} \, \eta^1 = \frac{\eta \, \nu}{k \, T} \quad . \tag{3.16}$$

According to Eq. (2.5), we have

$$\pi_1 = \frac{U \, c^3}{\nu^2 \, k \, T} \left(\frac{\eta \, \nu}{k \, T}\right)^N = \varphi_R(\pi_2) = \varphi_R\left(\frac{\eta \, \nu}{k \, T}\right) \quad , \tag{3.17}$$

where $\varphi_R(\eta \, \nu/(k \, T))$ is the universal function. Rearranging yields then

$$U(\nu, T) = \frac{\nu^2}{c^3} \, k \, T \left(\frac{\eta \, \nu}{k \, T}\right)^{-N} \varphi_R\left(\frac{\eta \, \nu}{k \, T}\right) \quad . \tag{3.18}$$

For historical reasons and convenience, we may introduce $8 \, \pi$ into Eq. (3.18). In doing so, we have

$$U(\nu, T) = \frac{8 \, \pi \, \nu^2}{c^3} \, k \, T \left(\frac{\eta \, \nu}{k \, T}\right)^{-N} \Phi_R\left(\frac{\eta \, \nu}{k \, T}\right) \quad , \tag{3.19}$$

where

$$\Phi_R\left(\frac{\eta \, \nu}{k \, T}\right) = \frac{1}{8 \, \pi} \, \varphi_R\left(\frac{\eta \, \nu}{k \, T}\right) \quad . \tag{3.20}$$

The merit of Eq. (3.18) is that the monochromatic energy density $U(\nu, T)$ is led to the universal function $\varphi_R(\eta \, \nu/(k \, T))$ which only depends on the non-dimensional argument $\pi_2 = \eta \, \nu/(k \, T)$. Obviously, this equation substantially agrees with Kirchhoff's [22] findings. Equation (3.19) may also be considered as a generalized form of Wien's [47] displacement law. The conventional form of this law can be derived by setting $N = -1$. In doing so, one obtains

$$U(\nu, T) = \frac{8 \, \pi \, \eta \, \nu^3}{c^3} \, \Phi_R\left(\frac{\eta \, \nu}{k \, T}\right) \quad . \tag{3.21}$$

3.2.1   Criteria for determining the universal function heuristically

As mentioned before, the universal function $\Phi_R(\eta \, \nu/(k \, T))$ cannot explicitly be inferred from dimensional arguments. However, any form of such a universal function must be compatible with following criteria: (a) There must exist a distinguish maximum. (b)It must guarantee that the integral in

$$E(T) = 8 \, \pi \, k \left(\frac{k}{c \, \eta}\right)^3 T^4 \int_0^\infty X^{2 \, - \, N} \Phi_R(X) \, dX \tag{3.22}$$

with $X = \pi_2 = \eta \, \nu/(k \, T)$ keeps finite. According to this equation, the constant a in Eq. (3.10) can be identified as

$$a = 8 \, \pi \, k \left(\frac{k}{c \, \eta}\right)^3 \int_0^\infty X^{2 \, - \, N} \Phi_R(X) \, dX \quad . \tag{3.23}$$

Note that Wien [48] also recognized the power law of Stefan [45] and Boltzmann [7] in deriving the quantity $c_1 \, \nu^3$ that occurs in his radiation law. First, it must fulfill the requirement that the integral in Eq. (3.23) is convergent. If we assume, for instance, $\Phi_R(X) = X^{N \, - \, 3}$, then we will obtain for this integral

$$\int_0^\infty X^{2 \, - \, N} \Phi_R(X) \, dX = \lim_{\substack{b \to \infty \\ a \to 0}} \int_a^b \frac{dX}{X} = \lim_{\substack{b \to \infty \\ a \to 0}} \ln \frac{b}{a} \quad . \tag{3.24}$$

Obviously, this integral is divergent, too. Consequently, the tendency of the integrand $X^{2 \, - \, N} \Phi_R(X)$ to zero when X approaches to infinity must be faster than X^{-1}. Ehrenfest [14], who only considered the case $N = -1$, postulated that, therefore, the requirement

$$\lim_{\nu \to \infty} X^4 \, \Phi_R(X) = 0 \tag{3.25a}$$

must be fulfilled. He coined it the *violet requirement* (also called the *violet condition* [24]). For any real number of $N < 3$ we may slightly modify it to

$$\lim_{\nu \to \infty} X^{3-N} \, \Phi_R(X) = 0 \quad . \tag{3.25b}$$

A more rigorous requirement, of course, would be

$$\lim_{\nu \to \infty} X^m \, \Phi_R(X) = 0 \quad , \tag{3.26}$$

where $m > 3 - N$. This means that for $X \to \infty$ the universal function $\Phi_R(X)$ must tend with a higher intense to zero than X^{-m} for any $m > 3 - N$. For $N = -1$, Ehrenfest [14] coined it the *strengthened violet requirement*. (c) The energy density, $U(\nu, T)$, must tend to unity when ν becomes smaller and smaller because the Rayleigh-Jeans law (3.8) is the asymptotic solution for that case. Following Ehrenfest [14], this requirement may be called the *red requirement* (also called the *red condition* [24]) being expressed by

$$\lim_{\nu \to 0} \left(\frac{\eta \, \nu}{k \, T} \right)^{-N} \Phi_R \left(\frac{\eta \, \nu}{k \, T} \right) = 1 \quad . \tag{3.27}$$

3.2.2   The Maximum condition

The first derivative of Eq. (3.19) reads

$$U'(\nu, T) = \frac{\alpha}{\beta^N} \, \nu^{1-N} \{ (2 - N) \, \Phi_R(\beta \, \nu) + \beta \, \nu \, \Phi_R'(\beta \, \nu) \} \quad , \tag{3.28}$$

with $\alpha = c_1 = 8 \, \pi \, \eta / c^3$ and $\beta = \eta / (k \, T)$. Thus we obtain for the extreme of $U(\nu, T)$

$$U'(\nu_e, T) = 0 \quad \leftrightarrow \quad (2 - N) \, \Phi_R(\beta \, \nu_e) + \beta \, \nu_e \, \Phi_R'(\beta \, \nu_e) = 0 \quad , \tag{3.29}$$

where ν_e is the frequency of the extreme. Defining $x = \beta \, \nu_e$ results in

$$(2 - N) \, \Phi_R(x) + x \, \Phi_R'(x) = 0 \quad . \tag{3.30}$$

The solution of the equation is given by

$$\Phi_R(x) = x^{N-2} \quad . \tag{3.31}$$

For ν_e and a given temperature $\Phi_R(x)$ provides a certain value Z. Thus, one obtains

$$\frac{\nu_e}{T} = \frac{k}{\eta} \, Z^{\frac{1}{N-2}} = \text{const.} \tag{3.32}$$

that may reflect Wien's displacement relation $\nu_{\max}/T = \text{const.}$ or $\lambda_{\max} \, T = const.$ Here, $\nu_{\max}$ is the frequency, for which $U(\nu, T)$ reaches its maximum, and $\lambda_{\max}$ is the corresponding wavelength. The second derivative reads

$$U''(\nu, T) = \frac{\alpha}{\beta^N} \, \nu^{-N} \, \{ (1 - N)(2 - N)\Phi_R(\beta \, \nu) + 2 \, (2 - N) \, \beta \, \nu \, \Phi_R'(\beta \, \nu) + (\beta \, \nu)^2 \, \Phi_R''(\beta \, \nu) \} \quad . \tag{3.33}$$

Considering Eq. (3.31) yields then for the extreme

$$U''(\nu_e, T) = 0 \quad , \tag{3.34}$$

i.e., we have a stationary point of inflection, rather than a maximum ($U''(\nu_e, T) < 0$) as requested. This is valid for any finite value of N, i.e., power laws as expressed by Eq. (3.31) are excluded due to the maximum condition. Since the maximum condition is not generally fulfilled, both Ehrenfest's [14] statement that Wien's displacement law does not impose any restriction a priori on the form of $\Phi_R(X)$ and Sommerfeld's

[43] derivation of Wien's displacement relation are not entirely accurate.

In contrast to Ehrenfest's statement, the condition $U"(\nu_e, T) < 0$ clearly imposes a restriction on the form of $\Phi_R(X)$. For $N < 2$, for instance, the exponential function,

$$\Phi_R(X) = \exp(-X) \quad, \tag{3.35}$$

guarantees that the maximum condition $U"(\nu_e, T) < 0$ is always fulfilled (see Appendix B).

3.2.3   Ehrenfest's red and violet requirements and the blackbody radiation laws of Rayleigh, Wien and Paschen, Thiesen as well as Planck

Using this exponential function (3.35), the integral in Eq. (3.23) becomes

$$\int_0^\infty X^{2-N} \exp(-X)\, dX = \Gamma(3-N) \quad. \tag{3.36}$$

Here, Euler's Γ-function defined by

$$\Gamma(\chi) = \int_0^\infty X^{\chi-1} \exp(-X)\, dX \tag{3.37}$$

for all real numbers $\chi = 3 - N > 0$ has been applied. This condition is clearly fulfilled when the restriction of the maximum condition, $N < 2$, is considered. Obviously, using this exponential function guarantees that the integral in Eq. (3.23) keeps finite. It also obeys the strengthened violet requirement. If we accept this exponential function for a moment, we will obtain

$$U(\nu, T) = \frac{8\,\pi\,\nu^2}{c^3}\, k\, T \left(\frac{\eta\,\nu}{k\,T}\right)^{-N} \exp\left(-\frac{\eta\,\nu}{k\,T}\right) \quad. \tag{3.38}$$

For $N = 0$, for instance, we obtain Rayleigh's [29] radiation formula. In this case the integral in Eq. (3.36) amounts to $\Gamma(3) = 2$. Choosing $N = -1$ yields the radiation law of Wien [48] and Paschen [36] with $\Gamma(4) = 6$. In the case of Thiesen's [46] radiation law, which can be derived by setting $N = -1/2$, we will obtain $\Gamma(3.5) = 3.3234$. Obviously, Eq. (3.38) contains the radiation laws of (a) Rayleigh, (b) Wien and Paschen, and (c) Thiesen as special cases.

Apparently, both the Wien-Paschen radiation law and that of Thiesen obey the strengthened violet requirement and, of course, fulfill the maximum condition. However, they do not tend to the classical blackbody radiation law of Rayleigh and Jeans given by Eq. (3.8), i.e., they do not obey the red requirement. If we namely express the exponential function by a Maclaurin series, we will obtain for the Wien-Paschen radiation law ($N = -1$)

$$exp(X) = 1 + X + \frac{X^2}{2} + \frac{X^3}{6} + \dots \tag{3.39}$$

For small values of X or $\eta\,\nu \ll k\,T$ this series can be approximated by

$$exp(X) \cong 1 + X \quad. \tag{3.40}$$

Thus, Eq. (3.38) may read

$$U(\nu, T) = \frac{8\,\pi\,\nu^2}{c^3}\, k\, T\, \frac{X}{1+X} \quad. \tag{3.41}$$

Obviously, the expression $X/(1+X)$ does not converge to unity, as requested by Eq. (3.8) (see also Figure 1). The same is true in the case of Thiesen's radiation law that can similarly be approximated for that range by

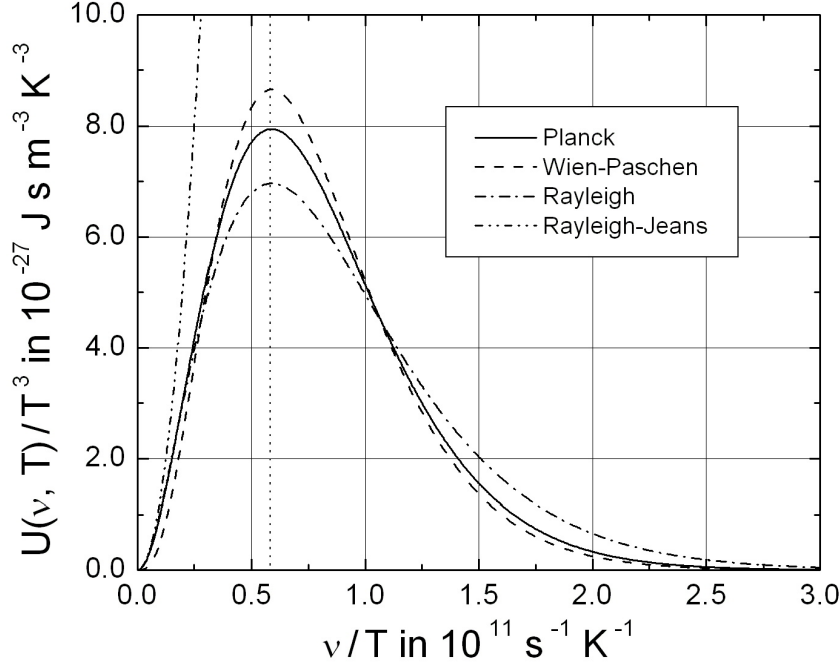$$U(\nu, T) = \frac{8\,\pi\,\nu^2}{c^3}\, k\, T\, \frac{X^{\frac{1}{2}}}{1+X} \quad. \tag{3.42}$$

Figure 1: Planck's [37, 38, 39] function of the spectral energy density. The curve represents $U(\nu, T)/T^3$ versus ν/T so that it becomes independent of T. Note that the constant, $C = 5.8787 \cdot 10^{10}\ s^{-1}\ K^{-1}$, occurring in Wien's displacement relationship $C = \nu_{\max}/T$ can be inferred from this figure (dotted line). Also shown are the functions of Wien [48], Paschen [36], Rayleigh [29], as well as Rayleigh [29, 30] and Jeans [20]. The constants used for plotting these functions are listed in Table 1.

On the contrary, if we choose

$$\Phi_R(X) = \frac{1}{\exp{(X)} - 1} \quad , \tag{3.43}$$

and, again, $N = -1$, Eq. (3.19) will provide

$$U(\nu, T) = \frac{8\,\pi\,\eta\,\nu^3}{c^3}\ \frac{1}{\exp\left(\dfrac{\eta\,\nu}{k\,T}\right) - 1} \quad . \tag{3.44}$$

This equation is quite similar to Planck's [37, 38, 39] radiation law. If we again express the exponential function by a Maclaurin series and again assume $\eta\,\nu \ll k\,T$, we will obtain

$$U(\nu, T) = \frac{8\,\pi\,\nu^2}{c^3}\ k\,T\ \frac{X}{1 + X - 1} = \frac{8\,\pi\,\nu^2}{c^3}\ k\,T \quad . \tag{3.45}$$

Apparently, Eq. (3.44) completely fulfils the red requirement (see Figure 1). For $\eta\,\nu \gg k\,T$, i.e., $\exp{(X)} \gg 1$, one also obtains the Wien-Paschen radiation law. This fact simply states that Eq. (3.44) also fulfils the strengthened violet requirement (see Figure 1).

Now, we have to check whether the integral in Eq. (3.23) keeps finite. Applying $\Phi_R(X) = (\exp{(X)} - 1)^{-1}$

Table 1: Quantities used for plotting the functions illustrated in Figures 1 and 2.

Author	k $(J\ K^{-1})$	η $(J\ s)$	C $(s^{-1}\ K^{-1})$
Planck	$1.3806 \cdot 10^{-23}$	$6.6262 \cdot 10^{-34}$	$5.8787 \cdot 10^{10}$
Wien − Paschen	$1.7963 \cdot 10^{-23}$	$9.1670 \cdot 10^{-34}$	same
Thiesen	$1.8768 \cdot 10^{-23}$	$7.9813 \cdot 10^{-34}$	same
Rayleigh	$1.5967 \cdot 10^{-23}$	$5.4323 \cdot 10^{-34}$	same
Rayleigh − Jeans	$1.3806 \cdot 10^{-23}$	$6.6262 \cdot 10^{-34}$	

and assuming $N = -1$ yield then

$$
\begin{aligned}
\int_0^\infty \frac{X^3}{\exp(X) - 1}\, dX &= \int_0^\infty \frac{X^3 \exp(-X)}{1 - \exp(-X)}\, dX \\
&= \int_0^\infty X^3 \exp(-X)(1 + \exp(-X) + \exp(-2X) + \ldots)\, dX \\
&= \int_0^\infty X^3 (\exp(-X) + \exp(-2X) + \exp(-3X) + \ldots)\, dX \Big\}
\end{aligned} \qquad (3.46)
$$

Substituting nX $(n = 1, 2, 3, \ldots)$ by Y yields then

$$
\int_0^\infty \frac{X^3}{\exp(X) - 1}\, dX = \sum_{n=1}^\infty \frac{1}{n^4} \int_0^\infty Y^3 \exp(-Y)\, dY = \Gamma(4) \sum_{n=1}^\infty \frac{1}{n^4} = \frac{\pi^4}{15} \quad , \qquad (3.47)
$$

where the sum in this equation

$$
\sum_{n=1}^\infty \frac{1}{n^{2k}} = \frac{2^{2k-1}\, \pi^{2k}}{(2k)!}\, B_k \quad , \qquad (3.48)
$$

can be calculated using the Bernoulli number for $k = 2$, namely $B_2 = 1/30$. Apparently, the integral (3.47) is convergent. As σ and, hence, $a = 4\,\sigma/c$ are known, the constant η can be determined. In doing so, one obtains: $\eta = 6.6262 \cdot 10^{-34}\ J\ s$, i.e., it equals the Planck constant h. Note that Planck called it the "Wirkungsquantum" that means an elementary quantum of action, and the product $h\,\nu$ is customarily designated as the *quantum of energy*. Furthermore, the expression

$$
\frac{1}{\exp\left(\dfrac{h\,\nu}{k\,T}\right) - 1} = \frac{1}{\exp\left(\dfrac{\hbar\,\omega}{k\,T}\right) - 1} \qquad (3.49)
$$

that occurs in Eq. (3.44) is customarily called the Planck distribution. It may be considered as a special case of the Bose-Einstein-distribution when the chemical potential of a "gas" of photons is considered as $\mu = 0$ [8, 16, 28].
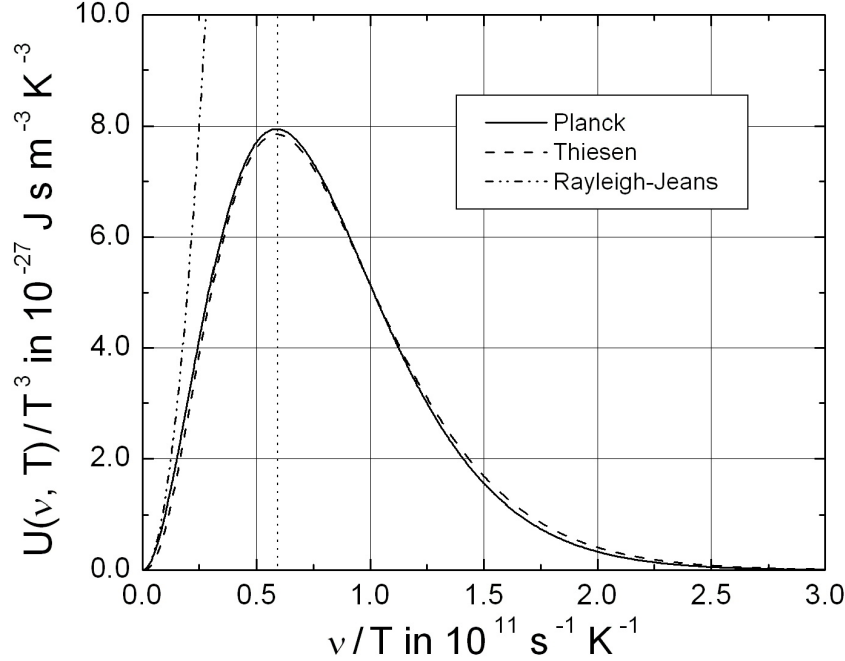
Figure 2: As in Figure 1, but the functions of Wien [48], Paschen [36], Rayleigh [29] are replaced by that of Thiesen [46].

It is obvious that Planck's [37, 38, 39] law obeys the requirements in the red range and in the violet range, and, in addition, it fulfils the maximum conditions. The same, of course, is also true in the case of Rayleigh's [29] radiation formula; but as illustrated in Figure 1 there are appreciable differences between both radiation laws in the higher-frequency range. These differences can also be inferred to the value of the integral in Eq. (3.36); for $N = 0$, it amounts to $\Gamma(3) = 2$. In that range also notable differences exist between Planck's law and those of Wien [48] and Paschen [36] because of similar reasons (see Figure 1). Since $\pi^4/15 \cong 6.49$, Planck's distribution yields a value of the integral in Eq. (3.23) which is notably larger than that provided by this equation when for the asymptotic solution for $h \nu \gg k T$, namely the radiation law of Wien and Paschen, $N = -1$, is chosen. As illustrated in Figure 2, the differences between the radiation laws of Planck and Thiesen [46] are appreciably smaller than in the other cases. These differences are so small that empirical results alone may be insufficient for evaluating both radiation laws. Since Thiesen's radiation law, of course, does not fulfill the red requirement, Planck's one has to be preferred. Even though the radiation law of Wien and Paschen represents an asymptotic solution for Planck's law, the former hardly represents a better solution than Thiesen's one. One may speculate that the small differences between the radiation laws of Planck and Thiesen would be ignored today. Note that the functions of Planck and Rayleigh-Jeans illustrated in Figure 1 are based on the values of $k = 1.3806 \cdot 10^{-23} \ J \ K^{-1}$ and $\eta = h = 6.6262 \cdot 10^{-34} J \ s$ currently recommended. The corresponding values for the functions of Wien, Thiesen, and Rayleigh have been derived using Stefan's constant $\sigma = 5.6696 \cdot 10^{-8} \ Jm^{-2} \ s^{-1}K^{-4}$ and Wien's relation $\nu_{\max}/T = const. = 5.8787 \cdot 10^{10} \ s^{-1} \ K^{-1}$. This constant was calculated using $X = 2.82144$, iteratively computed on the basis of the transcendental equation $X = 3 \ (1 - \exp(-X))$ obtained from Planck's radiation law as the condition for which $U(\nu, T)$ reaches its maximum [28], and the recommend values of k and h.

4   Some historical notes

Planck presented his blackbody radiation law at a meeting of the German Physical Society on October 19, 1900 in the form of [37, 28, 27, 34, 40],

$$U(\nu, T) = c_1 \, \frac{\nu^3}{\exp\left(c_2 \, \dfrac{\nu}{T}\right) - 1} \tag{4.1}$$

On the contrary, Eq. (3.44) reflects the form of Planck's radiation law[2] as presented in his seminal paper published at the beginning of 1901 [39] and presented to the German Physical Society on December 14, 1900 [38, 24, 27, 34, 40]. This day may be designated the *birthday of quantum theory* because the elementary quantum of action explicitly occurred in Planck's radiation law (e.g., [27, 34]. Our use of principles of dimensional analysis in heuristically deriving Eq. (3.44) by ignoring the aid of the linear harmonic oscillator model and Planck´s assumption that the energy occurring in Boltzmann´s [6] distribution is quantized gives evidence that Planck´s findings were the results of a lucky chance. In his Nobel Lecture, delivered in 1920, Planck objectively stated: "...even if the radiation formula should prove itself to be absolutely accurate, it would still only have, within the significance of a happily chosen interpolation formula, a strictly limited value."

   The first who, indeed, realized the true nature of Planck's constant was Einstein [15]. In his article he related the monochromatic radiation, from a thermodynamic point of view, to mutually independent light quanta (or photons) and their magnitude, in principle, to $\Delta\varepsilon = h\,\nu$ that occurs in Planck's radiation law (see Eq. (3.44)). Einstein, however, did not deal with Planck's radiation law in its exact manner, but rather with the approximation that fits the radiation law of Wien [48] and Paschen [36]. It seems that he first recognized that the quantum discontinuity was an essential part of Planck's radiation theory (e.g., [11, 25, 27]). As discussed by Klein [24] and Navarro and Pérez [33], a milestone on the road of quantum discontinuity and light quanta is Ehrenfest's [14] article on the essential nature of the different quantum hypotheses in radiation theory. With his article Ehrenfest contributed to the clarification of the hypothesis of light quanta. Unfortunately his contribution was not recognized for a long time.

Appendix A: Jeans' attempt to derive Wien's displacement law using dimensional analysis

Following Jeans [19, 21], the similarity hypothesis reads $F(Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Q_7, Q_8) = F(U, \lambda, T, c, e, m, R, K) = 0$. Here, $Q_2 = \lambda$ is the wavelength, $Q_5 = e$ and $Q_6 = m$ are the charge and the mass of an electron, respectively, $Q_7 = R$ is the universal gas constant, and $Q_8 = K$ the dielectric constant of the ether expressed with respect to an arbitrary measuring system. All other symbols have the same meaning as mentioned before. Since the monochromatic energy density has to be considered, the number of dimensional quantities is $\kappa = 8$ now. The dimensional matrix is given by

$$\mathbf{G} = \left\{ \begin{array}{cccccccc} -1 & 1 & 0 & 1 & 3/2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & -1 & -1 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 & 1/2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 1 \end{array} \right\} \tag{A1}$$

that can be inferred from the table of fundamental dimensions given by

---

[2] Note that Planck derived Eq. (3.44) using an equation for the entropy S of a linear harmonic oscillator, and he related it to its mean energy and to the quantum of energy, $h\,\nu$.

	U	λ	T	c	e	m	R	K
Length	- 1	1	0	1	3/2	0	2	0
Temperature	0	0	1	0	0	0	- 1	0
Time	- 1	0	0	- 1	- 1	0	- 2	0
Mass	1	0	0	0	1/2	1	1	0
Arbitrary system	0	0	0	0	1/2	0	0	1

The homogeneous system of linear equations can then be written as (see Eq. (2.12))

$$
\begin{Bmatrix}
-1 & 1 & 0 & 1 & 3/2 & 0 & 2 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\
-1 & 0 & 0 & -1 & -1 & 0 & -2 & 0 \\
1 & 0 & 0 & 0 & 1/2 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 1
\end{Bmatrix}
\begin{Bmatrix}
x_{1,i} \\ x_{2,i} \\ x_{3,i} \\ x_{4,i} \\ x_{5,i} \\ x_{6,i} \\ x_{7,i} \\ x_{8,i}
\end{Bmatrix}
= \{0\} \quad \text{for } i = 1, 2, 3 \tag{A2}
$$

Since the rank of the dimensional matrix is $r = 5$, we have $p = \kappa - r = 3$ non-dimensional π-invariants. Thus, a universal function of the form $\pi_1 = \varphi(\pi_2, \pi_3)$ is established. These three π-invariants can be derived from

$$
\left.
\begin{aligned}
-x_{1,i} \; + x_{2,i} \qquad\quad + x_{4,i} \; + 3/2\, x_{5,i} \qquad\qquad + 2\, x_{7,i} \qquad\quad &= 0 \\
x_{3,i} \qquad\qquad\qquad\qquad\qquad - x_{7,i} \qquad\quad &= 0 \\
-x_{1,i} \qquad\qquad\quad - x_{4,i} \quad - x_{5,i} \qquad\qquad - 2\, x_{7,i} \qquad\quad &= 0 \\
x_{1,i} \qquad\qquad\qquad\qquad + 1/2\, x_{5,i} \; + x_{6,i} \; + x_{7,i} \qquad\quad &= 0 \\
1/2\, x_{5,i} \qquad\qquad\qquad\quad + x_{8,i} &= 0
\end{aligned}
\right\} \quad \text{for } i = 1, 2, 3 \quad . \tag{A3}
$$

Choosing $x_{1,1} = 1$, $x_{6,1} = 0$, $x_{7,1} = -1$, $x_{1,2} = 0$, $x_{6,2} = -1$, $x_{7,2} = 1$, $x_{3,1} = 0$, $x_{6,3} = -2$ and $x_{7,3} = 1$ yields $x_{2,1} = 2$, $x_{3,1} = -1$, $x_{4,1} = 1$, $x_{5,1} = 0$, $x_{8,1} = 0$, $x_{2,2} = 0$, $x_{3,2} = 1$, $x_{4,2} = -2$, $x_{5,2} = 0$, $x_{8,2} = 0$, $x_{2,3} = -1$, $x_{3,3} = 1$, $x_{4,3} = -4$, $x_{5,3} = 2$, and $x_{8,3} = -1$. The choice $x_{1,1} = 1$, $x_{1,2} = 0$, and $x_{1,3} = 0$ is required to guarantee that $U(\lambda, T)$ only occurs explicitly. In accord with Eq. (2.2), the π-invariants are given by

$$
\pi_1 = \prod_{j=1}^{8} Q_j^{x_{j,1}} = U^1 \, \lambda^2 \, T^{-1} \, c^1 \, e^0 \, m^0 \, R^{-1} \, K^0 = \frac{U \, \lambda^2 \, c}{R \, T} \quad , \tag{A4}
$$

$$
\pi_2 = \prod_{j=1}^{8} Q_j^{x_{j,2}} = U^0 \, \lambda^0 \, T^1 \, c^{-2} \, e^0 \, m^{-1} \, R^1 \, K^0 = \frac{R \, T}{m \, c^2} \quad , \tag{A5}
$$

and

$$
\pi_3 = \prod_{j=1}^{8} Q_j^{x_{j,3}} = U^0 \, \lambda^{-1} \, T^1 \, c^{-4} \, e^2 \, m^{-2} \, R^1 \, K^{-1} = \frac{R \, T \, e^2}{\lambda \, m^2 \, c^4 \, K} \quad . \tag{A6}
$$

According to Eq. (2.5), we have

$$
\pi_1 = \frac{U \, \lambda^2 \, c}{R \, T} = \varphi\left\{ \frac{R \, T}{m \, c^2}, \frac{R \, T \, e^2}{\lambda \, m^2 \, c^4 \, K} \right\} \tag{A7}
$$

or

$$
U(\lambda, T) = \frac{R \, T}{\lambda^2 \, c} \, \varphi\left\{ \frac{R \, T}{m \, c^2}, \frac{R \, T \, e^2}{\lambda \, m^2 \, c^4 \, K} \right\} \quad . \tag{A8}
$$

It is apparent that Eq. (A8) completely disagrees with

$$U(\lambda, T) \propto \frac{T}{\lambda^4} \, \varphi_R(\lambda, T) \tag{A9}$$

or

$$U(\lambda, T) \propto \lambda^{-5} \, \varphi_R(\lambda, T) \tag{A10}$$

that can be derived from Eqs. (3.19) and (3.21), respectively. Since we always have $x_{2,i} = 2 \, x_{1,i}$, any other choice can give no better relationships. Therefore, Jeans' attempt to derive Wien's [47] displacement law completely fails because of his inadequate similarity hypothesis.

Appendix B: Derivation of Wien's displacement relation for $\Phi_R(X) = \exp(-X)$

If we consider the exponential function $\Phi_R(\beta \, \nu) = \exp(-\beta \, \nu)$, the first two derivatives of Eq. (3.19) will read

$$U'(\nu, T) = \frac{\alpha}{\beta^N} \, \nu^{1-N} \, (2 - N - \beta \, \nu) \, \exp(-\beta \, \nu) \tag{B1}$$

and

$$U''(\nu, T) = \frac{\alpha}{\beta^N} \, \nu^{-N} \, \{(1-N)(2-N) - 2 \, (2-N) \, \beta \, \nu + (\beta \, \nu)^2\} \, \exp(-\beta \, \nu) \quad . \tag{B2}$$

For the extreme we obtain $\beta \, \nu_e = 2 - N$ or

$$\frac{\nu_e}{T} = (2 - N) \, \frac{k}{\eta} = \text{const.} \tag{B3}$$

Introducing Wien's displacement relation into Eq. (B2) yields finally

$$U''(\nu_e, T) = (N - 2) \, \frac{\alpha}{\beta^N} \, \exp(N - 2) \quad , \tag{B4}$$

This means that $U''(\nu_e, T)$ becomes negative for any real value of N that fulfils the condition $N < 2$, i.e., in such case the condition of a maximum is fulfilled.

References

[1]   Barenblatt, G.I., 1979: Similarity, Self-Similarity, and Intermediate Asymptotics. Consultants Bureau, New York/London, 218 pp.

[2]   Barenblatt, G.I., 1994: Scaling Phenomena in Fluid Mechanics. Inaugural Lecture delivered before the University of Cambridge, May 3, 1993. Cambridge University Press, Cambridge, U.K., 45 pp.

[3]   Barenblatt, G.I., 1996: Similarity, Self-Similarity, and Intermediate Asymptotics. Cambridge University Press, Cambridge, U.K., 386 pp.

[4]   Barenblatt, G.I., 2003: Scaling. Cambridge University Press, Cambridge, U.K., 171 pp.

[5]   Bohren, C.F., 2004: Dimensional analysis, falling bodies, and the fine art of not solving differential equations. Am. J. Phys. 72, 534-537.

[6] Boltzmann, L., 1877:"Über die Beziehung zwischen dem zweiten Hauptsatz der mechanischen Wärmetheorie und der Wahrscheinlichkeitsrechnung respektive den Sätzen über das Wärmegleichgewicht", Wien Ber. 76, p373, Wiss. Abh. II, p. 164 (in German).

[7] Boltzmann, L., 1884: Ableitung des Stefan'schen Gesetzes, betreffend die Abhängigkeit der Wärmestrahlung von der Temperatur aus der electromagnetischen Lichttheorie". Wiedemann's Annalen 22, 291-294 (in German).

[8] Bose, S.N., 1924: Plancks Gesetz und Lichtquantenhypothese. Z. Phys. 26, 178-181 (in German).

[9] Brown, R.A., 1991: Fluid Mechanics of the Atmosphere. Academic Press, San Diego, CA, 489 pp.

[10] Buckingham, E., 1914: On physically similar systems; illustrations of the use of dimensional equations. Physical Review 4, 345-376.

[11] Darrigol, O., 1992: From c-Numbers to q-numbers: The Classical Analogy in the History of Quantum Theory. University of California Press, Berkeley, CA.

[12] Ehrenfest, P., 1906a: Bemerkungen zu einer neuen Ableitung des Wienschen Verschiebungsgesetztes. Phys. Z. 7, 527-528 (in German).

[13] Ehrenfest, P., 1906b: Bemerkungen zu einer neuen Ableitung des Wienschen Verschiebungsgesetztes (Antwort auf Herrn Jeans' Entgegnung). Phys. Z. 7, 850-852 (in German).

[14] Ehrenfest, P., 1911: Welche Züge der Lichtquantenhypothese spielen in der Theorie der Wärmestrahlung eine wesentliche Rolle. Ann. d. Physik 36, 91-118 (in German).

[15] Einstein, A., 1905: Über einen die Erzeugung und Verwandlung des Lichtes betreffenden heuristischen Gesichtspunkt. Ann. d. Physik 17, 164-181 (in German).

[16] Einstein, A., 1924/25: Quantentheorie des einatomigen idealen Gases", Sitzungsber. Preuß. Akad. Wiss., Berlin (1924), 262-267; "Zweite Abhandlung", Sitzungsber. Preuß. Akad. Wiss., Berlin (1925), 3-14; "Quantentheorie des idealen Gases", Sitzungsber. Preuß. Akad. Wiss., Berlin (1925), 18-25 (in German; as cited by Rechenberg, 1995).

[17] Herbert, F., 1980: Vorlesung zur Physik der planetarischen Grenzschicht. Vorlesungskriptum, J.W. Goethe-Universität, Frankfurt/Main, Germany.

[18] Huntley, H.E., 1952: Dimensional Analysis. MacDonald & Co., Publishers, London, U.K., 158 pp.

[19] Jeans, J.H., 1905a: On the laws of radiation. Proc. Royal Soc. A 78, 546-567.

[20] Jeans, J. H., 1905b: A comparison between two theories of radiation", Nature 72 (1905), 293-294.

[21] Jeans, J.H., 1906: Bemerkungen zu einer neuen Ableitung des Wienschen Verschiebungsgesetztes. Erwiderung auf Herrn P. Ehrenfests Abhandlung. Phys. Z. 7, p. 667 (in German).

[22] Kirchhoff, G., 1860: Ueber das Verhältniss zwischen dem Emissionsvermögen und dem Absorptionsvermögen für Wärme und Licht. Ann. d. Physik u. Chemie 109, 275-301 (in German).

[23] Kitaigorodskij, S.A., 1976: Zur Anwendung der Ähnlichkeitstheorie für die Beschreibung der Turbulenz in der bodennahen Schicht der Atmosphäre. Z. Meteorologie 26, 185-205 (in German).

[24] Klein, M.J., 1970: Paul Ehrenfest - Volume 1: The Making of a Theoretical Physicist. North-Holland Publishing Comp., Amsterdam/London, 330 pp.

[25] Krah, H., 1999: Quantum Generations: A History of Physics in the Twentieth Century. Princeton University Press.

[26]   Kramm, G. and Herbert, F. 2006: The structure functions for velocity and temperature fields from the perspective of dimensional Scaling. Flow, Turbulence, and Combustion 76, 23-60.

[27]   Kuhn, T.S., 1978: Black-Body Theory and the Quantum Discontinuity 1894-1912. Oxford University Press, Oxford/New York, 356 pp.

[28]   Landau, L.D. and Lifshitz, E.M., 1975: Lehrbuch der theoretischen Physik - Vol V: Statistische Physik. Akademie-Verlag Berlin, 527 pp. (in German).

[29]   Lord Rayleigh (J.W. Strutt), 1900: Remarks upon the law of complete radiation. Phil. Mag. 49, 539-540.

[30]   Lord Rayleigh, 1905: The dynamical theory of gases and of radiation. Nature 72, 54-55.

[31]   Lorentz, H. A., 1903: On the emission and absorption by metals of rays of heat of great wave-length. Proc. Acad. Amsterdam 5, 666-685.

[32]   Lummer, O. and Pringsheim, E., 1900: Über die Strahlung des schwarzen Körpers für lange Wellen", Verh. d. Deutsch. Phys. Ges. 2, p. 163.

[33]   Navarro, L. and Pérez, E., 2004: Paul Ehrenfest on the necessity of Quanta (1911): Discontinuity, quantization, corpuscularity and adiabatic invariance. Arch. Hist. Exact Sci. 58, 97-141.

[34]   Pais, A., 1995: Introducing Atoms and their Nuclei. In: L.M. Brown, A. Pais, and Sir B. Pippard (eds.), Twentieth Century Physics - Vol. I. Institute of Physics Publishing, Bristol and Philadelphia, and American Institute of Physics Press, New York, pp. 43-141.

[35]   Pal Arya, S., 1988: Introduction to Micrometeorology. Academic Press, San Diego, CA, 303 pp.

[36]   Paschen, F., 1896: Ueber die Gesetzmäßigkeiten in den Spektren fester Körper. Ann. d. Physik 58, 455-492 (in German).

[37]   Planck, M., 1900a: Über eine Verbesserung der Wien'schen Spektralgleichung. Verh. d. Deutsch. Phys. Ges. 2, 202-204 (in German).

[38]   Planck, M., 1900b: Zur Theorie des Gesetzes der Energieverteilung im Normalspectrum", Verh. d. Deutsch. Phys. Ges. 2, 237-245 (in German).

[39]   Planck, M., 1901: Ueber das Gesetz der Energieverteilung im Normalspectrum. Ann. d. Physik 4, 553-563 (in German).

[40]   Rechenberg, H., 1995: Quanta and Quantum Mechanics. In: L.M. Brown, A. Pais, and Sir B. Pippard (eds.), Twentieth Century Physics - Vol. I. Institute of Physics Publishing, Bristol and Philadelphia, and American Institute of Physics Press, New York, pp. 143-248.

[41]   Rubens, H. and Kurlbaum, R., 1900: Über die Emission langwelliger Wärmestrahlen durch den schwarzen Körper bei verschiedenen Temperaturen. Sitzungsber. d. K. Akad. d. Wissensch. zu Berlin vom 25. Oktober 1900, p. 929-941 (in German).

[42]   Rubens, H. and Kurlbaum, R., 1901: Anwendung der Methode der Reststrahlen zur Prüfung des Strahlungsgesetzes. Ann. d. Physik 4, 649-666 (in German).

[43]   Sommerfeld, A., 1956: Thermodynamics and Statistical Mechanics", Lectures on Theoretical Physics, Vol. V. Academic Press Inc., Publishers, New York, 401 pp.

[44]   Sorbjan, Z., 1989: Structure of the Atmospheric Boundary Layer. Prentice Hall, Englewood Cliffs, NJ., 317 pp.

[45] Stefan, J., 1879: Über die Beziehung zwischen der Wärmestrahlung und der Temperatur. Wiener Ber. II, 79, 391-428 (in German).

[46] Thiesen, M. F., 1900: Über das Gesetz der schwarzen Strahlung. Verh. d. Deutsch. Phys. Ges. 2, 65-70 (in German).

[47] Wien, W., 1894: Temperatur und Entropie der Strahlung. Ann. d. Physik 52, 132-165 (in German).

[48] Wien, W., 1896: Ueber die Energieverteilung im Emissionsspectrum eines schwarzen Körpers. Ann. d. Physik 58, 662-669 (in German).

[49] Zdunkowski, W. and Bott, A., 2003: Dynamics of the Atmosphere. Cambridge University Press, Cambridge, U.K., 719 pp.